ISSN : 2248-9622, Vol. 4, Issue 1(Version 1), January 2014, pp.116-121

RESEARCH ARTICLE OPEN ACCESS

Trajectory Optimization of Launch Vehicles Using Steepest Descent Method – A Novel Approach

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ABSTRACT
Trajectory optimization of a generic launch vehicle is considered in this paper. The direct application of a nonlinear programming method is used in recent literature, which transforms the original problem into a nonlinear optimization problem.To study the rocket motion under the influence of gravitational field, 2-D simulator is developed. The rocket motion is analyzed for a gravity turn trajectory. The objective is to ensure desired terminal conditions as well as minimum control effort in the low dynamic pressure region. Design of optimal trajectory for a single stage rocket is a two point boundary problem. Trajectory is designed for a single stage liquid rocket.Trajectory is computed for a given initial and final condition using equations of motion of rocket in 2-D plane. Hamiltonian is formulated for the given constraints. The non-linear equations are solved using steepest descent method.It is assumed that the launch vehicle is not experiencing any perturbations. Results are compared for Runge-kutta and Euler's integration methods,which clearly brings out the potential advantages of the proposed approach.
Keywords – Trajectory optimization, Steepest descent method, Euler's method, Runge-Kutta method
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I. INTRODUCTION

The subject of optimization of a continuous dynamical system has a long and interesting history. The first example is the Brachistochrone problem posed by Galileo, later by Bernoulli and solved by Newton in 1696. The problem can be simply stated as the determination of a trajectory that satisfies specified initial and terminal conditions, i.e., satisfies the system governing equations, while minimizing some quantity of importance. We use the term trajectory here as representing a path or time history of the system state variables. Our experience is primarily in the field of spacecraft and aircraft trajectory optimization so that the trajectories are literal [1].There are many techniques for numerically solving trajectory optimization problems. Generally these techniques are classified as either indirect or direct. Indirect methods are characterized by explicitly solving the optimality conditions stated in terms of the adjoint differential equations, the Pontryagin's maximum principle, and associated boundary conditions. Using the calculus of variations, the optimal control necessary conditions can be derived by setting the first variation of the Hamiltonian function zero. The indirect approach usually requires the solution of a nonlinear multipoint boundary value problem. There is a comprehensive survey paper by Betts [8] that describes direct and indirect optimization, the relation between these two approaches, and the development of these two approaches. In it, Betts points out some of the disadvantages with indirect methods which are mentioned below First, it is necessary to derive analytic expressions for the necessary conditions, and for complicated nonlinear dynamics this can become quite daunting. Second, the region of convergence for a root finding algorithm may be surprisingly small, especially when it is necessary to guess values for the adjoint variables that may not have an obvious physical interpretation. Third, for problems with path inequalities it is necessary to guess the sequence of constrained and unconstrained subarcs before iteration can begin[8]. One of the standard procedures for optimizing non-linear system is the Gradient or Steepest-decent technique. Reference [3] discusses implementation of this method to a launch vehicle carrying a hypersonic vehicle as payload. In paper[7] the BDH method that is one of the direct collocation methods is used. In the direct collocation method, not only the control variables but also the state variables are discretised. The BDH is using linear interpolation for this discretization.An optimization algorithm Combination of Gauss Pseudospectral Method and Genetic Algorithm is presented to solve the optimal finite-thrust trajectory with an input constraint in the paper[9]. The simulation results indicate that the GPM-GA optimization algorithm has high accuracy, and the error with results solved by indirect method is very small. In paper[10], two different approaches are proposed for simultaneous optimization of staging and trajectory of multistage launch vehicles. In the first approach, the problems of staging optimization and trajectory optimization are solved separately. In the second approach, the optimal staging and trajectory are achieved during trajectory optimization in an integrated problem. Both approaches can lead to very similar solutions in spite of their differences in staging formulation. Integrated approach can lead to better results because of simultaneous consideration of objective functions and effective constraints of two optimization problems of staging and trajectory. In paper[11] analysis of Euler approximation to a state constrained control problem is carried out. it shows that if the active constraints satisfy an independence condition and the Lagrangian satisfies a coercively condition, then locally there exists a solution to the Euler discretization, and the error is bounded by a constant times the mesh size. The paper[12] analyze second-order Runge-Kutta approximations to a nonlinear optimal control problem with control constraints. If the optimal control has a derivative of bounded variation and a coercively condition holds, and it shows that for a special class of Runge-Kutta schemes, the error in the discrete approximating control is O(h2), where h is the mesh spacing.In this paper the trajectory optimization problem solved by steepest descent method, which is the type of indirect gradient method. In reference[6], the control variable is randomly chosen which will affect the accuracy to a great extent. This problem is solved in our paper, by carefully choosing the initial control variable by an new approach which is described in the following sections. To the author's best knowledge, the tuning of weighting is not done in any of the available literature. In this the papers the weighting factor is selected by proper tuning, and the proposed method is explained in the following sections. Also for the quick convergence of the objective function a multiplication factor is introduced. This will drastically increase the performance of the proposed system. The prescribed optimization technique is implemented using Euler's and Runge-kutta[2] integration methods and the performance is verified.

1.1. Trajectory optimization problem

A general problem statement for finite-thrust trajectory optimization can be stated as follows [7]. A trajectory optimization or optimal control problem[14]can be formulated as a collection of$N$phases. In general, the independent variable t(time) for phase $k$is defined in the region $t_0(k) \leq t \geq t_f(k)$ for many applications, and the phases are sequential, that is

$$t_0(k+1) = t_f(k) \qquad (1)$$

However, neither of those assumptions is required. Within phase $k$the dynamics of the system are described by a set of dynamic variables.

$$Z = \begin{bmatrix} y_k(t) \\ u_k(t) \end{bmatrix} \qquad (2)$$

Made up of the$n_y(k)$state variables and the $n_u(k)$control variables, respectively. In addition, the dynamics may incorporate the $n_p(k)$parameters p^k which are not dependent on$t$. For clarity the phase-dependent is dropped notation from the remaining discussion in thissection. However, it is important to remember that many complex problem descriptions require different dynamics and/or constraints, and a phase-dependent formulation accommodates this requirement.

Typically the dynamics of the system are defined by a set of ordinary differential equations written in explicit form, which are referred to as the state or system equations,

$$\dot{y} = f[(y(t), u(t), p, t)] \qquad (3)$$

Where$y$is the$n_y$dimension state vector.

Initial conditions at time t_0 are defined by,

$$\Psi_{0l} \leq \Psi[y(t_0), u(t_0), p, t_0] \leq \Psi_{0u} \qquad (4)$$

Where,

$$\Psi[y(t_0), u(t_0), p, t_0] = \Psi_{0u} \qquad (5)$$

And terminal conditions at the final time$t_f$are defined by,

$$\Psi_{fl} \leq \Psi[y(t_f), u(t_f), p, t_f] \leq \Psi_{fu} \qquad (6)$$

Where,

$$\Psi[y(t_f), u(t_f), p, t_f] = \Psi_{fu} \qquad (7)$$

In addition, the solution must satisfy algebraic path constraints of the form,

$$g_{fl} \leq g[y(t_f), u(t_f), p, t_f] \geq g_{fu} \qquad (8)$$

Where $g$is a vector of sizen_g, as well as simple bounds on the state variables.

$$y_1 \leq y(t) \leq y_u \qquad (9)$$

And control variables,

$$u_1 \leq u(t) \leq u_u \qquad (10)$$

Note that an equality constraint can be imposed if the upper and lower bounds are equal,

$$(g_l)k = (g_u)k \qquad (11)$$

for some *k*.

Finally, it may be convenient to evaluate expressions of the form,

$$\int_{t0}^{tf} q[y(t), u(t), p, t]\, dt \qquad (12)$$

Which involve the quadrature functions q collectively we refer to those functions evaluated during the phase, namely,

$$F(t) = \begin{bmatrix} f[y(t), u(t), p, t] \\ g[y(t), u(t), p, t] \\ q[y(t), u(t), p, t] \end{bmatrix} \qquad (13)$$

As the vector of continuous functions, similarly functions evaluated at a specific point, such as the boundary conditions $\Psi\,[y\,(t_0)\,, u(t_0)\,, t_0)\,]$ and $\Psi\,[y\,(t_f)\,, u(t_f)\,, t_f]$ are referred to as point functions.

The basic optimal control problem is to determine the *k* dimensional control vectors $u^{(k)}(t)$and parameters $p^{(k)}$to minimize the performance index.

$$J = \Phi[y(t_0^{(1)}), t_0^1, y(t_f^{(1)}), p^{(1)}, t_f^1 \cdots\cdots\cdots, \quad y(t_0^{(N)}), t_0^N, y(t_f^{(N)}), p^{(N)}, t_f^N] \qquad (14)$$

Notice that the objective function may depend on quantities equations of motion with non-rotating spherical earth[5] is given by,

$$\dot{r} = V sin\gamma \qquad (15)$$

$$\dot{V} = \frac{1}{mV}(Tcos\alpha - D - mgsin\gamma) \qquad (16)$$

$$\dot{\gamma} = \frac{1}{mV}(Tsin\alpha + L) + \left(\frac{V}{r} - \frac{g}{V}\right) cos\gamma \qquad (17)$$

Here neglecting $drag, D$ and $lift, L$ since the rocket propagating in above vacuum, therefore the equations $(15), (16) \& (17)$ becomes

$$\dot{r} = V sin\gamma \qquad (18)$$

$$\dot{V} = \frac{1}{mV}(Tcos\alpha - mgsin\gamma) \qquad (19)$$

$$\dot{\gamma} = \frac{1}{mV}(Tsin\alpha) + \left(\frac{V}{r} - \frac{g}{V}\right) cos\gamma \qquad (20)$$

Trajectory optimization problem is formulated as optimal control problem. The nonlinear problem has to solve using numerical method. Steepest decent method is used to solve the problem.

1.2 Euler's method

Euler method is a first-order numerical procedure for solving ordinary differential equations (ODEs) with a given initial value. The Euler method is a first-order method, which means that the local error (error per step) is proportional to the square of the step size, and the global error (error at a given time) is proportional to the step size. It also suffers from stability problems.Euler's method can be implemented both in simulator program and the opimal control problem. In simulator program the integration of $r, v and \gamma$ happens. That is given as follows.

$$r = r_0 + \dot{r}dt \qquad (21)$$

$$v = v_0 + \dot{v}dt \qquad (22)$$

$$\gamma = \gamma_0 + \dot{\gamma}dt \qquad (23)$$

In opimal control the $\lambda^T$is integrating to get final λ values.

$$\lambda_1 = \lambda_{10} + \dot{\lambda_1}dt \qquad (24)$$

$$\lambda_2 = \lambda_{20} + \dot{\lambda_2}dt \qquad (25)$$

$$\lambda_3 = \lambda_{30} + \dot{\lambda_3}dt \qquad (26)$$

Where $dt$is the integration step size.$r_0, v_0 \& \gamma_0$ are the initial values of $r, v and \gamma$. Similarly $\lambda_{10}, \lambda_{20} \& \lambda_{30}$are the initial values of $\lambda_1, \lambda_2 \& \lambda_3$.For optimal solutions, The optimality conditionis given by,

$$\frac{\partial H}{\partial \alpha} = 0 \qquad (27)$$

1.3 Runge-Kutta method

Runge-Kutta methods[7] are very popular because of their efficiency; and are used for in most solving problem numerically. They are single-step methods, as the Euler methods[7].

There are many ways to evaluate the right-hand side $f(x, y)$ that all agree to first order, but that have different coefficients of higher-order error terms. Adding up the right combination of these, it can eliminate the error terms order by order. That is the basic idea of the Runge-Kutta method.

$$k_1 = hf(x_n, y_n) \qquad (28)$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right) \qquad (29)$$

$$k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right) \qquad (30)$$

$$k_4 = hf(x_n + h, y_n + k_3) \qquad (31)$$

The fourth-order Runge-Kutta method requires four evaluations of the right-hand side.

Methodology

In this section mathematical model of rocket trajectory is discussed first. To study the rocket motion under the influence of gravitational field, 2- D simulator is developed. The rocket motion is analyzed for a gravity turn trajectory. It is solved by Newton-Raphson method. Using gravity turn trajectory target is achieved by varying initial conditions to reach the target. From this we will get the characteristics of the required trajectory. Indirect methods require initial guesses for the control as well as adjoint variables.

For finding the feasible steering profile, acceleration is assumed to be a linearly increasing quantity. So rate of change of velocity is approximated as linear function.This linear function is selected by polynomial approximation[16] that is by changing the functions randomly to fit the curve properly.

$$c_1 t + c_2 = \dot{V} \quad (32)$$

$$c_1 t + c_2 = \frac{1}{mV}(T cos\alpha - mg sin\gamma) \quad (33)$$

$$cos\alpha = \frac{c_1 t + c_2 + g sin\gamma}{T/m} \quad (34)$$

$$\alpha = cos^{-1}\left(\frac{c_1 t + c_2 + g sin\gamma}{T/m}\right) \quad (35)$$

The vehicle acceleration is a linear function of time as given in the equation (32)the constants are obtained by varying initial and final steering angle value. Thus tuning of constantsc_1 and $c_2$is done. The initial and final α is thus obtained to be 3.5 deg and 14 deg respectively from boundary conditions.

Therefore by solving the equation (35) using the initial and final values the appropriate values for c_1 and c_2 can be found out;

$$c_1 = 6.4217 \times 10^{-3}, c_2 = 1.918235$$

A simulator program for rocket trajectory is then developed by putting the initial guess for control variable...The target can be successfully achieved only with t continuous monitoring of the position, velocity and flight path angle of the rocket. The state and control vectors are defined as,

$$\dot{X} = [x_1\ x_2\ x_3]^T \quad (36)$$

$$\dot{X} = [r\ v\ \gamma]^T \quad (37)$$

$$U = \alpha \quad (38)$$

In this a trajectory optimization problem of a single stage launch vehicle is considered. The objective here is to generate the guidance command history$\alpha(t), t \in (t_0, t_f)$such that the following concerns are takencare of,

(1) At the final timet_f, the specified terminal constraintsmust meet accurately. The terminal constraints include constraints on altitude, velocity and flight path angle (which is the angle made by the velocity vector with respect to the local horizontal).

(2) The system should demand minimum guidance command, which can be ensured by formulating a 'minimum time' problem.

To achieve the above objectives, the following cost function is selected, which consists of terminal penalty terms and a dynamic control minimization term.

$$J = (r(t_f) - r_f)^2 s_r + (v(t_f) - v_f)^2 s_v + (row(t_f) - row_f)^2 s_\gamma \quad (39)$$

An optimal control problem is developed for rocket trajectory and control parameter is optimized by steepest descent method. The method of steepest descent will generally converge linearly to the solution [14, 15]. The initial values for ad-joint variables are calculated by manual calculations using the terminal conditions of rocket.

For optimal solutions, the optimality condition is given by,

$$\frac{\partial H}{\partial \alpha} = 0 \quad (40)$$

The weighting factor τ should be selected as follows:-

$$tow = \frac{\partial H}{\partial \alpha} \times \left(\frac{\partial H}{\partial \alpha}\right)^T \quad (41)$$

$$\tau = \frac{1}{\sqrt[2]{tow}} \quad (42)$$

Thus weighting factor,τ thus obtained is applied to the control variableα ,to update the initial alpha profile.

$$\alpha = \alpha_0 - \frac{\partial H}{\partial \alpha} \times \tau \quad (43)$$

Where α_0 = previous stage of α

Equation (43) is the gradiant function which optimizes the control variable.

In the above equation,multiplication factor,s has to be addedto converge objective function value. The value has to multiplied with the weighting factor,τ.

$$\alpha = \alpha_0 - \frac{\partial H}{\partial \alpha} \times \tau \times s \quad (44)$$

1.4 Summary

The steps given below are adopted from [6]

1. Start with an initial guess of control variable$\alpha_0(t)$ where $t_0 \le t \le t_f$
2. Propagate the states from t_0 to $t_f$using $\alpha^k(t)$ with initial conditions X_0(Forward Integration of the SystemDynamics).
3. Obtain $\lambda_k(t_f)$ by using the boundary conditions (terminal boundary conditions).
4. Propagate the costate vector from t_f to $t_0$using step (iii) values as the initial values (Backward integration ofCostate Equations).
5. Calculate the gradient $\left[\frac{\partial H}{\partial \alpha}\right]$ from t_0 to $t_f - \Delta t$
6. Calculate the control update $\alpha^{(k+1)}(t) - \tau\frac{\partial H}{\partial \alpha} \times s$where $\tau \epsilon (0,1)$is the learningrate.
7. Repeat from step (2) to step (6) until optimality conditions are meet within a specified tolerance.

Table.1 EULER'S METHOD

Parametres	Intialvalue	Final Value	Target Value	Error Obtained
Height	428901.5	485462.57	485462.59	−0.022
Velocity	5890.112	7623.539 m/s	7623.5321	0.0075
Flightpath Angle	5.2229 d	−0.9904	0.002 deg	0.9906

Final objective function value=8.4609×10^{-6}

Total number of iterations=81

I. Result analysis

The results obtained by Euler's method and Runge-Kutta methods are given in table 1 and 2.Runge-kutta and Euler's methods of integration are performed. In Runge-kutta method of integration, the step size is different for each integration step. But in Euler's method the steps sizes are equal. Due to this Runge-Kutta method of integration is comparatively accurate.

Figure(1) shows the variations in control variable with respect to time. Since the initial value of the control variable is a guess, it is not optimal, hence rate of change of initial control variable is high and therefore it requires more fuel to complete the trajectory.

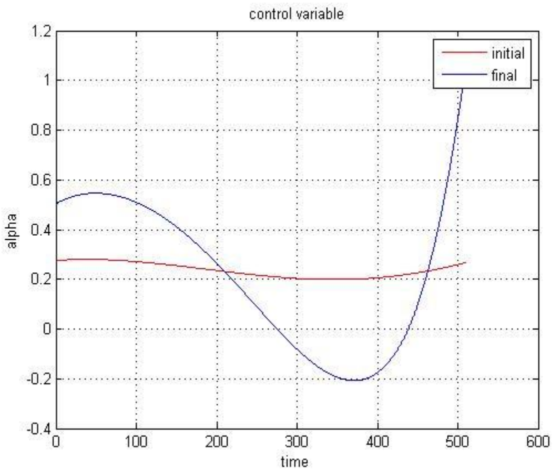

Fig.1: Initial and final control variable array

Figure(2) shows the variation of objective function in two different methods. From this graph we can find that the convergence characteristics of two methods are same but Runge-Kutta method needs more iteration to find its minima then also it is efficient by calculating the efficiency. Figure(3) shows the variation in flight path angle with respect to time we can see that the performance in achieving target in both methods are almost same.

Table.2RUNGE-KUTTA METHOD

Parametres	Intial value	Final Value	Target Value	Error Obtained
Height	42890	485462.5	485462.59	−0.0003
Velocity	5890.1	7623.532	7623.5321	−0.0001
Flightpath Angle	5.2229	−0.9896	0.002 deg	−0.9898

Final objective function value=2.9858×10^{-6}

Total number of iterations =116

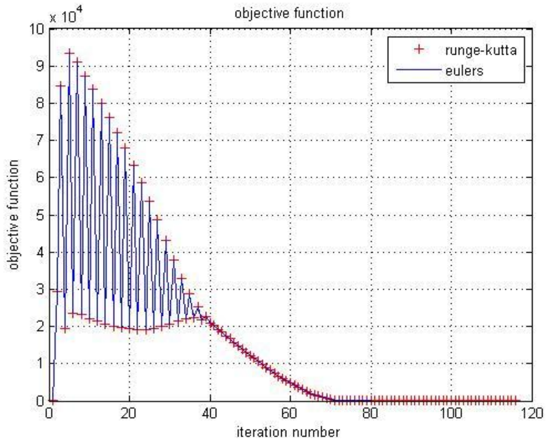

Fig.2 Objective function variations in Runge-Kutta and Euler's method

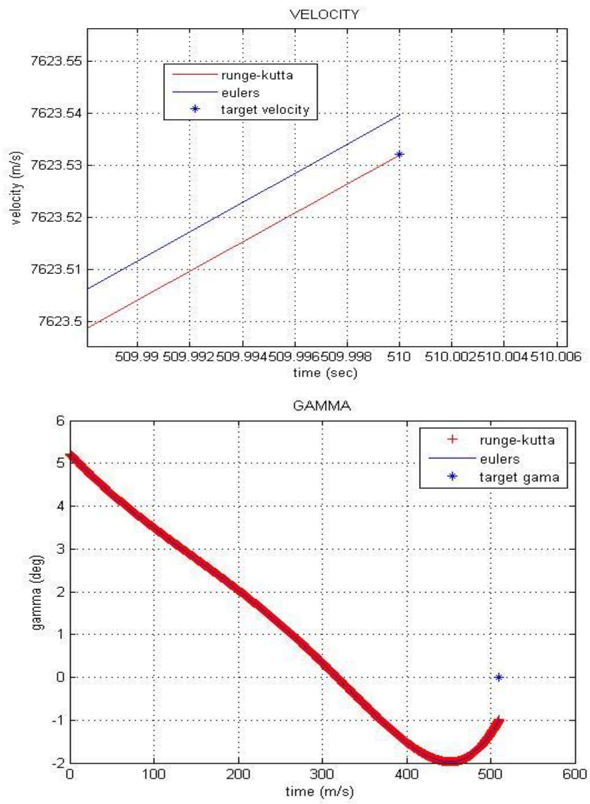

Fig.3 Final value of velocity difference in Runge-kutta method and Euler's method

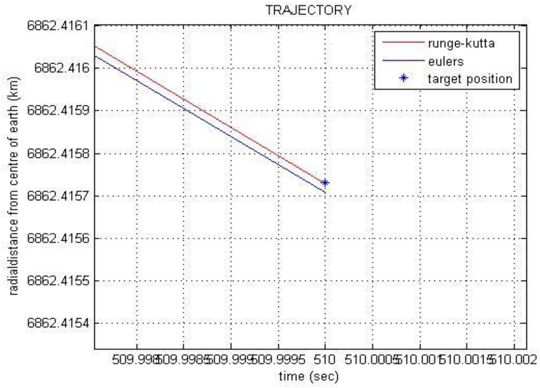

Fig.4 Final position achieved by rocket using Runge-kutta method and Euler's method

Figure(4) and (5) indicates the velocity profile and position with respect to time and by analyzing both graphs we can estimate the efficiency of Runge-Kutta method.

II. Conclusion

Rocket trajectories are optimized to achieve the target, by either minimum time, control forces or fuel. To study the rocket motion under the influence of gravitational field, 2-D simulator is developed. The rocket motion is analyzed for a gravity turn trajectory and target is achieved by varying initial conditions to reach the target. Design of optimal trajectory for a single stage rocket is a two point boundary problem. Trajectory is designed for a single stage liquid rocket, for given initial and final conditions using equations of motion of rocket in 2-D plane. The trajectory optimized for minimum time of flight using the pitch angle as control variable.Hamiltonian is formulated for the given constraints. The non-linear equation is solved using steepest descent method.Results are compared for Runge-kutta integration and Euler integration method.Runge-kutta integration gives high accuracy. By analyzing the results the converging characteristics of two methods are almost same but Runge-Kutta method needsmore time to settle down to the minimum value.

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